Biometrical Letters
Vol. 48 (2011), No. 1, 33-39

Some efficiencies of two-way elimination of heterogeneity designs

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Summary

Simple lower bounds for A-, D-, E- and L-efficiency of some two-way elimination of heterogeneity designs are derived. The bounds are obtained for treatment effects on the basis of the eigenvalues of information matrix **C** with respect to the diagonal matrix **R**.

Key words: A-efficiency, D-efficiency; E-efficiency, L-efficiency, eigenvalues, information matrix, lower bound, two-way elimination of heterogeneity design.

1. Introduction

Any arrangement of v treatments in b_1 rows and b_2 columns is called a two-way elimination of heterogeneity design. Let $\mathbf{r} = (r_1, ..., r_v)'$, $\mathbf{k}_1 = (k_{1_1}, ..., k_{1_{b_1}})'$ and $\mathbf{k}_2 = (k_{2_1}, ..., k_{2_{b_2}})'$ denote a vector of treatment replications, a vector of row sizes and a vector of column sizes, respectively. Let $\mathbf{R}$, $\mathbf{K}_1$ and $\mathbf{K}_2$ be the diagonal matrices with the successive elements of $\mathbf{r}$, $\mathbf{k}_1$ and $\mathbf{k}_2$ on their diagonals. Moreover, let $\mathbf{N}_1$ be the $v \times b_1$ treatment-row incidence matrix, let $\mathbf{N}_2$ be the $v \times b_2$ treatment-column incidence matrix. The $\mathbf{C}$-matrices of the two related subdesigns are

$$\mathbf{C}_s = \mathbf{R} - \mathbf{N}_s \mathbf{K}_s^{-1} \mathbf{N}_s' \qquad (1)$$

with $s = 1$ for the treatment-row subdesign and $s = 2$ for the treatment-column subdesign.

In this paper we consider designs with information matrix for the treatment effects defined by Berube and Styan (1993):

$$\mathbf{C} = \xi_1 \mathbf{C}_1 + \xi_2 \mathbf{C}_2 - \xi_0 \mathbf{C}_0, \qquad (2)$$

where $\xi_1 > 0$, $\xi_2 > 0$, and $\xi_0 > 0$, $\mathbf{C}_0 = \mathbf{R} - \mathbf{r}\mathbf{r}'/n$ and n is the number of experimental units. Let $D\left(n, v, b_1, b_2, r_{min}, r_{max}, k_{1_{max}}, k_{2_{max}}, h\right)$ denote the set of two-way elimination of heterogeneity designs whose **C**-matrix admit a representation in the form (2), where $r_{min} = \min\limits_{1\leq i\leq v} r_i$, $r_{max} = \max\limits_{1\leq i\leq v} r_i$, $k_{1_{max}} = \max\limits_{1\leq j\leq b_1} k_{1_j}$, $k_{2_{max}} = \max\limits_{1\leq j\leq b_2} k_{2_j}$, and h is the rank of **C** ($h \leq v-1$, if $h = v-1$ then a design is said to be connected).

It should be noted that in the theory of experimental designs, A-, D- and E-optimality is often considered. For example, Filipiak and Szczepańska (2005) and Moerbeek (2005) considered A-, D- and E-optimality for designs for quadratic and cubic growth curve models and for designs for polynomial growth models with auto-correlated errors, respectively. A-optimal chemical balance weighing designs and A-optimal designs under a quadratic growth curve model in the transformed time interval are presented respectively by Ceranka *et al.* (2007) and Filipiak and Szczepańska (2007). The E-optimality of some two-way elimination of heterogeneity designs, of nested row-column designs, of designs in irregular BIB settings, of designs with three treatments and of designs under an interference model is considered by Kozłowska and Walkowiak (1990a), Brzeskwiniewicz (1995), Bagchi (1996), Morgan and Reck (2007) and Filipiak and Różański (2005), respectively. Note that A-, D-, E- and L-efficiency for block designs is described by Brzeskwiniewicz (1996).

2. Results

For a design $d \in D(n, v, b_1, b_2, r_{min}, r_{max}, k_{1_{max}}, k_{2_{max}}, h)$ let $0 = \epsilon_{d_0} \leq \epsilon_{d_1} \leq \ldots \leq \epsilon_{d_{v-1}} \leq 1$ be eigenvalues of its **C**-matrix with respect to the matrix **R**. Define

$$\begin{aligned} &\phi_{A|R}(d) = \textstyle\sum_{i=v-h}^{v-1} \epsilon_{d_i}^{-1}, \quad &&\phi_{D|R}(d) = \textstyle\prod_{i=v-h}^{v-1} \epsilon_{d_i}^{-1}, \\ &\phi_{E|R}(d) = \epsilon_{d_{v-h}}, &&\phi_{L|R}(d) = \textstyle\sum_{i=v-h}^{v-1} \epsilon_{d_i}. \end{aligned} \tag{3}$$

A design d is A- or D-optimal if it minimizes the values $\phi_{A|R}(d)$ or $\phi_{D|R}(d)$ among all those possible from some class of designs. A design d is E- or L-optimal if it maximizes the values $\phi_{E|R}(d)$ or $\phi_{L|R}(d)$ among all those possible from some class of designs. The A-, D-, E- and L-efficiency of a design d is defined to be

$$\begin{aligned} e_{A|R}(d) &= \frac{\phi_{A|R}(d_A^*)}{\phi_{A|R}(d)}, \qquad & e_{D|R}(d) &= \frac{\phi_{D|R}(d_D^*)}{\phi_{D|R}(d)}, \\ e_{E|R}(d) &= \frac{\phi_{E|R}(d)}{\phi_{E|R}(d_E^*)}, & e_{L|R}(d) &= \frac{\phi_{L|R}(d)}{\phi_{L|R}(d_L^*)}, \end{aligned} \tag{4}$$

where d_A^*, d_D^*, d_E^* and d_L^* are A-, D-, E- and L-optimal designs, respectively.

One problem with these definitions is that optimal designs are known only for some special cases. Therefore, in the next section simple lower bounds of (4) will be given as some measures of the efficiencies of design d. First let us assume that $\xi_1 + \xi_2 - \xi_0 \leq 1$.

2.1. Lower bounds of $e_{A/R}$ and $e_{D/R}$

Note that for $d \in D\left(n, v, b_1, b_2, r_{min}, r_{max}, k_{1_{max}}, k_{2_{max}}, h\right)$ from (1) and (2) we have

$$\epsilon_{d_{v-1}} = \mathbf{p}'\mathbf{R}^{-1}\mathbf{C}\mathbf{p} = \xi_1\mathbf{p}'\mathbf{p} + \xi_2\mathbf{p}'\mathbf{p} -$$

$$-\xi_1\mathbf{p}'\mathbf{R}^{-1}\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'\mathbf{p} - \xi_2\mathbf{p}'\mathbf{R}^{-1}\mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2'\mathbf{p} - \xi_0\mathbf{p}'\mathbf{p} +$$

$$+\xi_0\mathbf{p}'\mathbf{R}^{-1}\tfrac{\mathbf{r}\mathbf{r}'}{n}\mathbf{p} \leq \xi_1 + \xi_2 - \xi_0 + \xi_0\mathbf{p}'\tfrac{\mathbf{1}\mathbf{r}'}{n}\mathbf{p} = \xi_1 + \xi_2 - \xi_0$$

because $\mathbf{p}'\mathbf{p} = 1$ and $\mathbf{p}'\mathbf{1} = 0$, where $\mathbf{p}$ is the eigenvector of matrix $\mathbf{R}^{-1}\mathbf{C}$. From above and (3) we have

$$\phi_{A|R}\left(d_A^*\right) \geq \frac{h}{\xi_1 + \xi_2 - \xi_0} \quad \text{and} \quad \phi_{D|R}\left(d_D^*\right) \geq \frac{1}{\left(\xi_1 + \xi_2 - \xi_0\right)^h}. \tag{5}$$

Next, observe that $tr\left(\mathbf{R}^{-1}\mathbf{C}\right) = \sum_{i=v-h}^{v-1} \epsilon_{d_i} \leq h$. In many cases a different method of estimation can be used, namely from (1) and (2) we have

$$\begin{aligned} & tr\left(\mathbf{R}^{-1}\mathbf{C}\right) = tr\left(\mathbf{R}^{-1}\left(\xi_1\mathbf{C}_1 + \xi_2\mathbf{C}_2 - \xi_0\mathbf{C}_0\right)\right) = \\ & \xi_1 \sum_{i=1}^{v}\left(1 - \sum_{j=1}^{b_1}\frac{n_{1_{ij}}^2}{r_i k_{1_j}}\right) + \xi_2 \sum_{i=1}^{v}\left(1 - \sum_{j=1}^{b_2}\frac{n_{2_{ij}}^2}{r_i k_{2_j}}\right) - \xi_0 \sum_{i=1}^{v}\left(1 - \frac{\mathbf{1}'\mathbf{r}}{n}\right) \leq \\ & \xi_1\left(v - \sum_{i=1}^{v}\frac{1}{r_i k_{1_{\max}}}\sum_{j=1}^{b_1} n_{d_{ij}}\right) + \xi_2\left(v - \sum_{i=1}^{v}\frac{1}{r_i k_{2_{\max}}}\sum_{j=1}^{b_2} n_{d_{ij}}\right) = \\ & \xi_1\frac{v(k_{1_{\max}}-1)}{k_{1_{\max}}} + \xi_2\frac{v(k_{2_{\max}}-1)}{k_{2_{\max}}} = t, \end{aligned} \tag{6}$$

because $\mathbf{1}'\mathbf{r} = n$ and $\sum_{j=1}^{b_1} n_{1_{ij}} = \sum_{j=1}^{b_2} n_{2_{ij}} = r_i$.

Note that

$$\bar{\epsilon}_d = \frac{\sum_{i=v-h}^{v-1} \epsilon_{d_i}}{h} \leq \frac{t}{h} \tag{7}$$

and

$$\sum_{i=v-h}^{v-1} \epsilon_{d_i}^{-1} \geq \tfrac{h}{\bar{\epsilon}_d} \quad \text{and} \quad \prod_{i=v-h}^{v-1} \epsilon_{d_i}^{-1} \geq \tfrac{1}{\bar{\epsilon}_d^h} \tag{8}$$

From (7) and (8) we have, in particular,

$$\phi_{A|R}\,(d_A^*) \geq \frac{h^2}{t} \quad \text{and} \quad \phi_{D|R}\,(d_D^*) \geq \left(\frac{h}{t}\right)^h. \tag{9}$$

From (5) and (9) follows that

$$phi_{A|R}\,(d_A^*) \geq \max\left\{\tfrac{h}{\xi_1+\xi_2-\xi_0},\ \tfrac{h^2}{t}\right\},$$

$$phi_{D|R}\,(d_D^*) \geq \max\left\{\tfrac{1}{(\xi_1+\xi_2-\xi_0)^h},\ \left(\tfrac{h}{t}\right)^h\right\},$$

which leads (see (4)) to

$$e_{A|R}(d) \geq \frac{\max\left\{\tfrac{h}{\xi_1+\xi_2-\xi_0},\ \tfrac{h^2}{t}\right\}}{\phi_{A|R}(d)}, \quad e_{D|R}(d) \geq \frac{\max\left\{\tfrac{h}{\xi_1+\xi_2-\xi_0},\ \left(\tfrac{h}{t}\right)^h\right\}}{\phi_{D|R}(d)} \tag{10}$$

and therefore two efficiency lower bounds of e_A and e_D are defined as

$$e'_{A|R}(d) = \frac{\max\left\{\tfrac{h}{\xi_1+\xi_2-\xi_0},\ \tfrac{h^2}{t}\right\}}{\phi_{A|R}(d)}, \quad e'_{D|R}(d) = \frac{\max\left\{\tfrac{1}{(\xi_1+\xi_2-\xi_0)^h},\ \left(\tfrac{h}{t}\right)^h\right\}}{\phi_{D|R}(d)}. \tag{11}$$

We have so far considered two-way elimination of heterogeneity designs fulfilling the condition $\xi_1 + \xi_2 - \xi_0 \leq 1$. There exist designs where the inequality $\xi_1 + \xi_2 - \xi_0 > 1$ is satisfied. For those designs the efficiency lower bounds of e_A and e_D are defined as

$$e'_{A|R}(d) = \frac{\max\left\{h,\ \tfrac{h^2}{t}\right\}}{\phi_{A|R}(d)}, \quad e'_{D|R}(d) = \frac{\max\left\{1,\ \left(\tfrac{h}{t}\right)^h\right\}}{\phi_{D|R}(d)}. \tag{12}$$

2.2. Lower bounds of $e_{E/R}$

Let row and column designs d_s, $s = 1{,}2$ with information matrix $\mathbf{C}_s$ (see (1)) contain a row (a column) which consists of m common distinct treatments and $2 \leq m \leq v-1$. We assume, by relabelling the treatments and reshuffling the row (column) as necessary, that the first row and column consists of m distinct treatments with numbers $1, ..., m$ and the first row size is k_{1_1} and the first column size is k_{2_1}. Then

$$\epsilon_{d_1} \leq \frac{v}{m(v-m)} \left(\xi_1 P_{d_1}(m) + \xi_2 P_{d_2}(m) - \xi_0 P_{d_0}(m)\right) = P_d(m), \qquad (13)$$

where $P_{d_s}(m) = \frac{\sum_{i=1}^m r_i}{r_{\min}}\left(1 - \frac{1}{k_{s_{\max}}}\right) - \frac{k_{s_1}-1}{r_{\max}}$, $s = 1, 2$ and the principal minor of $\mathbf{C}_0$ is at least $P_{d_0}(m) = m\left(1 - \frac{m r_{\max}^2}{n \cdot r_{\min}}\right)$, because $\sum_{i=1}^m r_i - \frac{1}{n}\sum_{i,j=1}^m r_i r_j \leq \sum_{i=1}^m r_i - \frac{(m r_{\max})^2}{n}$. Note that in the paper of Brzeskwiniewicz (1995) we have weak equality $P_{d_0}(m) = \sum_{i=1}^m r_i - \frac{\left(\sum_{i=1}^m r_i\right)^2}{n}$. On the other hand

$$\epsilon_{d_1} \leq \frac{v}{v-1}\left(\xi_1 T_{d_1} + \xi_2 T_{d_2} - \xi_0 T_{d_0}\right) = T_d, \qquad (14)$$

where $T_{d_s} = 1 - \frac{r_{min}}{r_{\max} k_{s_{\max}}}$ (Brzeskwiniewicz (1995)) and $T_{d_0} = 1 - \frac{r_{\max}}{n}$ because the i-th diagonal element of $\mathbf{C}_0$ is equal to $r_i - \frac{r_i^2}{n}$, and $r_i - \frac{r_i^2}{n} = r_i\left(1 - \frac{r_i}{n}\right) \leq r_{\max}\left(1 - \frac{r_{\min}}{n}\right)$. Note that in the paper of Brzeskwiniewicz (1995) we have weak equality $T_{d_0} = r_{\max}\left(1 - \frac{r_{\min}}{n}\right)$.
From (13) and (14) we have

$$\phi_{E|R}\left(d_E^*\right) \leq \min\{P_d(m),\ T_d\}. \qquad (15)$$

Observe that from (15) and (4) it follows that

$$e_{E|R}\left(d\right) \geq \frac{\phi_{E|R}(d)}{\min\{P_d(m),\ T_d\}} \qquad (16)$$

and therefore the lower bound of e_E is defined as

$$e'_{E|R}\left(d\right) = \frac{\phi_{E|R}(d)}{\min\{P_d(m),\ T_d\}}. \qquad (17)$$

2.3. Lower bounds of $e_{L/R}$

From (3) and (6) we have

$$\phi_{L|R}\,(d_L^*) \leq t. \tag{18}$$

Formulae (18) and (4) imply that

$$e_{L|R}(d) \geq \frac{\phi_{L|R}(d)}{t} \tag{19}$$

and therefore the lower bound of e_L is defined as

$$e'_{L|R}(d) = \frac{\phi_{L|R}(d)}{t}. \tag{20}$$

3. Examples

We consider the A-, D-, E- and L-efficiency of the designs shown in Tables 1 and 2.

Table 1.

	Columns			
Rows	1	2	3	4
1	1	2	4	3
2	7	8	5	6
3	5	6	1	2
4	3	4	8	7

Table 2.

	Columns						
Rows	1	2	3	4	5	6	7
1		3	5		2		
2			4	6		3	
3				5	7		4
4	5				6	1	
5		6				7	2
6	3		7				1
7	2	4		1			

In the case of Table 1, $d \in D(16, 8, 4, 4, 2, 2, 4, 4, 7)$ with $\xi_1 = \xi_2 = \xi_0 = 1$ and $\epsilon_{d_1} = \epsilon_{d_2} = \epsilon_{d_3} = \epsilon_{d_4} = \frac{1}{2}$, $\epsilon_{d_5} = \epsilon_{d_6} = \epsilon_{d_7} = 1$ Kozłowska and Walkowiak (1990b). We calculate $\phi_{\cdot}(d)$ occurring in (3) as: $\phi_{A|R}(d) = 11$, $\phi_{D|R}(d) = 16$, $\phi_{E|R}(d) = \frac{1}{2}$ and $\phi_{L|R}(d) = 5$. But d_1 and d_2 have no block with m distinct treatments, thus we calculate only T_d occurring in (13) as $T_d = \frac{5}{7}$. Hence according to formulae (11), (17) and (18) we obtain: $e'_{A|R}(d) = \frac{7}{11}$, $e'_{D|R}(d) = \frac{1}{16}$, $e'_{E|R}(d) = 0.7$ and $e'_{L|R}(d) = \frac{5}{12}$. We have obtained a high $e'_E(d)$ value, therefore we consider that this design is close to an E-optimal design, but is far from being an A-, D- and L-optimal design.

In Table 2, $d \in D(21,7,7,7,3,3,3,3,6)$ with $\xi_1 + \xi_2 = 1$, $\xi_0 = \frac{4}{9}$ and $\epsilon_{d_1} = \epsilon_{d_2} = \epsilon_{d_3} = \epsilon_{d_4} = \epsilon_{d_5} = \epsilon_{d_6} = 1$ (Agrawal (1966)). From (3) we have: $\phi_{A|R}(d) = 18$, $\phi_{D|R}(d) = 3^6$, $\phi_{E|R}(d) = \frac{1}{3}$ and $\phi_{L|R}(d) = 2$. But d_1 and d_2 have a block with $m = 3$ distinct treatments, thus we calculate the $P_d(3)$ and T_d occurring in (13) and (14), respectively; $P_d(3) = T_d = \frac{1}{3}$. From (11), (17) and (20) we obtain: $e'_{A|R}(d) = \frac{3}{5}$, $e'_{D|R}(d) = (\frac{3}{5})^6$, $e'_{E|R}(d) = 1$ and $e'_{L|R}(d) = \frac{3}{7}$. This design is far from being an A-, D- and L-optimal design, but it is an E-optimal design ($e'_E(d) = 1$).

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